

Fourier and Wavelet Analysis of Clifford-Valued Functions



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I hereby certify that the work embodied in this thesis has been done in collaboration with my supervisor, Dr. Jeff Hogan, and contains work published in peer-review papers of which I am a joint author. I have included as part of this thesis the following written statement, endorsed by my supervisor, attesting to my contribution to the joint publications.

During the writing of this thesis I have received advice, guidance, and mathematical assistance from my supervisor. His assistance has been within the scope of normal supervisor-student relations. Apart from his help, this thesis has been all my own work. Some of the results in this thesis are contained in the papers [36; 37], written jointly with my supervisor. The names on the papers are in alphabetical order, to conform with the usual convention in Pure Mathematics.

..... Date:

Andrew Joel Morris

..... Date:

Dr. Jeff Hogan

To Kate,

You make my world a more beautiful place!

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Abstract

Fourier analysis has long been studied as a method to analyse real-valued or complex-valued signals. The Clifford-Fourier transform recently developed by Brackx, De Schepper, and Sommen in [4] and [5] has led to the development of Fourier analytic methods for hypercomplex or Clifford-valued signals. In the quaternionic case, Brackx et al. have found the kernel of the Quaternionic Fourier transform which allows for much easier calculation, and we focus much of our attention in this thesis on the quaternionic case.

We define the continuous wavelet transform of quaternion-valued signals on the plane and prove a Calderón reproducing formula. We also define the monogenic signal, a generalization of the analytic signal of a function on the real line. We provide a characterization of translation-invariant operators and submodules of the quaternionic L_2 module. We develop several fundamental analogues of classical orthogonal wavelet theory pioneered by Cohen, Daubechies, Mallat, and Meyer to quaternion-valued functions on the plane. We include design conditions required to produce wavelets which have compact support and desired regularity. We also develop the basic theory needed for constructing a biorthogonal wavelet basis and construct an example.

For a general Clifford algebra, we develop a condition on f so that $f * g$ satisfies a convolution theorem. We also develop a Clifford-Fourier characterization of the Clifford-valued Hardy spaces on $\mathbb{R}^d$.